Cyclostationarity and stochastic resonance in threshold devices

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This paper intends to show how the theory of stochastic cyclostationary processes can be used to study stochastic resonance in static nonlinearities. The statistic we use is the covariance function of the output. The covariance is a second-order cumulant and is not dependent on by the mean. Furthermore, this covariance is not averaged in time as is usually done in the stochastic resonance literature. A two-dimensional Fourier transform of the covariance gives the so-called spectral correlation. The spectral correlation depends on the usual harmonic frequency and on another frequency, called cycle frequency. The cyclostationarity of a signal makes the spectral correlation discrete in the cycle frequency. The zero cycle frequency corresponds to the usual ''stationary power spectrum'' used in the stochastic resonance literature. We thus exploit all the second-order statistical information. We first revisit classical stochastic resonance in threshold devices using the spectral correlation, showing that the effect is seen for nonzero cycle frequencies. The cases of additive and multiplicative noise are detailed. We then study stochastic resonance in threshold devices for communication signals. These signals are usually modeled as stochastic cyclostationary processes. We show that stochastic resonance occurs, and the phenomenon is quantified using the spectral correlation of the output: The amplitude of the spectral correlation at nonzero cycle frequencies presents a maximum as the power of the input noise is increased. [S1063-651X(99)08405-6]

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I. INTRODUCTION

Stochastic resonance is usually described as a nonlinear phenomenon that allows an improved transmission of a signal by interaction with noise. In the past 15 years, this effect has been the subject of much research, both theoretical and experimental. The main theories of stochastic resonance as well as experimental evidence can be found in many reviews [1–4].

Several theories exist that describe stochastic resonance in dynamical systems. Unfortunately, since the problem is difficult, only approximate theories have been developed (e.g., adiabatic theory [5], linear response theory [1–4], etc.) However, for static nonlinearities, for which stochastic resonance has been reported quite recently [6–8], exact results may be written in certain circumstances, such as whiteness of the noise.

For that case, recent papers by Chapeau-Blondeau and Godivier [9,10] proposed a systematic treatment of stochastic resonance for periodic signals corrupted by additive white noise. Their theory is based on the fact that the output of the nonlinearity may be viewed as a periodic signal plus a perturbation. Using the Fourier series of the periodic signal and the second-order statistics of the perturbation, they were able to define clearly the signal-to-noise ratio at the harmonics of the output. Evidence of stochastic resonance was reported, and their theory was successfully confronted with an experiment.

Another interesting effect of stochastic resonance concerns ''aperiodic'' signals, such as harmonic noise, spike trains in neurons [11,12]. The effect of stochastic resonance was first quantified using an input-output coherence measure [12], and later by information theory tools [13–16]. For example, it is shown that the capacity (in the information theory sense) of a static nonlinear channel presents a stochastic resonance effect. But the capacity of a channel is also an ''input-output'' measure, since it represents the maximum of the so-called transinformation or mutual entropy. In these works, the signal is called aperiodic. However, this terminology is a little bit confusing, since signals are often considered as stochastic. They are in fact often cyclostationary: Their statistics are periodic with respect to the reference time [17].

The property of cyclostationarity is common to almost all works concerning stochastic resonance. But surprisingly, it has not been taken into account, since most of the works reported in the literature eliminate cyclostationarity by performing a coherent average in time of the statistics of interest. This average is equivalent to a ''stationarization'' by imposing on the signal a random uniformly distributed phase at the time origin.

However, cyclostationary signals are recognized to be of great importance to the signal processing community [18]. This is explained by the great number of engineering applications where cyclostationary signals occur: rotating machines, communication systems, and in general any applications where a clock (which can be hidden) rules the phenomena.

The aim of this paper is thus to explore the question of whether cyclostationary signals can lead to stochastic resonance effects when they are corrupted by noise and then nonlinearly transformed. Furthermore, we study static nonlinearities. The case of dynamical systems is under investigation.

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We will present in Sec. II the tools needed to manipulate cyclostationary stochastic processes. These tools are not different from those classically used (covariances), but they will be totally exploited in the rest of the paper, contrary to the literature where they are averaged to handle stationary quantities. The main tool we use is the so-called *spectral correlation,* which is the two-dimensional Fourier transform of the covariance function. The spectral correlation is then a two-dimensional function. The first variable is the traditional frequency, whereas the second one is called the *cycle frequency*. When a process is cyclostationary, its covariance function is a periodic (or almost periodic) function of the reference time. In that case, the cycle frequency takes only discrete values, revealing the periodicity of the signal analyzed. The spectral correlation at the zero cycle frequency (infinite period in the signal) is equivalent to the ''stationarized'' (or averaged) spectrum that is used in the stochastic resonance (SR) literature. In this section, we also plead for the use of the covariance function instead of the correlation function. The covariance is a *cumulant* based statistic, and therefore only retains second-order statistical features (it eliminates the contribution of the mean).

In Sec. III, we revisit classical stochastic resonance in static nonlinearities within the framework of cyclostationary processes. Stochastic resonance is studied for deterministic periodic signals corrupted by additive noise as well as multiplicative noise. We show that the output of a threshold device attacked by such signals is cyclostationary. Furthermore, we show that stochastic resonance is revealed by examining the spectral correlation amplitude at nonzero cycle frequencies. Working with cumulants rather than moments leads to strange results: For example, in the spectrum of an output signal, the usual peaks superimposed on the background noise spectrum disappear. This is due to the canceling of the mean in the cumulant calculation. In addition, this shows that results described in the SR literature concern the evaluation of the effect on the first-order statistics.
After showing how cyclostationarity can be used to quantify classical stochastic resonance, we turn to the problem of determining if SR can occur for cyclostationary stochastic processes. In Sec. IV, we examine signals that are widely used in communication systems. Basically, these signals are made of sequences of letters that are randomly taken into a discrete alphabet. Each letter is supported by a function of duration *T*. Hence, the time axis is cut into pieces of length *T*, each of which supports a letter. The signals are zero mean and cyclostationary: Their statistics are periodic in the reference time. Since many communication systems have nonlinearities such as threshold devices, we study what happens to a noisy communication signal when it passes through a static nonlinearity. Exact calculations are performed on some examples, and stochastic resonance is revealed in the spectral correlation of the output. Furthermore, we show that looking at only the stationarized spectrum may not reveal the effect, whereas considering nonzero cycle frequency always reveals the effect.

To conclude the paper, we make some remarks on the work developed here and give some ideas for future work.

II. BASICS ON ALMOST-CYCLOSTATIONARY SIGNALS

Authors working on stochastic resonance have of course noted that the signals involved are nonstationary, since their statistics are in general not invariant under a shift of the origin of time. In fact, statistics involved in stochastic resonance are often periodic in time [17,18]. In order to define the power spectrum of the signals, an average over one period is now usually made. This average has the effect of ''stationarizing'' the signals under consideration. Indeed, it can be shown that this operation is equivalent to imposing on the signals a random initial phase, uniformly distributed on a period. The aim of this section is to show that this operation is ''*ad hoc*,'' and that performing it creates a loss of information. To discuss this point, we now present (or recall) some facts on the theory of cyclostationary stochastic processes.

Almost cyclostationarity

Roughly speaking, a function $f(t)$ is almost periodic if it accepts a so-called Fourier-Bohr decomposition, $f(t) = \Sigma c_k \exp(2i\pi f_k t)$ where the sum is made over the set $A = \{f_k \text{ such that } c_k \neq 0\}$. This set is finite or infinite but countable. Furthermore, the frequencies involved are not in general a multiple of a fundamental harmonic. Coefficients c_k are obtained via $c_k = \lim_{T\to+\infty}(1/T)\int_0^T f(t)\exp(-2i\pi f_k t)dt$. Of course, periodic functions form a subclass of almost-periodic functions (in that case, the Fourier-Bohr decomposition reduces to the Fourier series expansion).

Let x_t be a stochastic process. This process is said to be strictly almost cyclostationary if the joint probability density function (PDF) $\{p_{x_t,x_{k+\tau_1},\ldots,x_{t+\tau_n}}(x_0,x_1,\ldots,x_n;t,\tau_1,\ldots,\tau_n)\}$ is an almost periodic function of *t*, and this is for all integers *n* and all lags τ_i. This implies that the statistics of the process are almost periodic with respect to *t*.

Let $M_x(t) = E[x_t]$ and $\Gamma_x(t,\tau) = \mathrm{Cov}[x_t, x_{t+\tau}]$ be the mean and the covariance function of x_t, respectively ($E[\,]$ stands for the mathematical expectation or set average, and Cov[ ] is the covariance operator). If x_t is almost cyclostationary, then its mean is in general almost periodic, and its covariance almost periodic with respect to *t*. If the mean and the covariance are only periodic, we will say that the process is cyclostationary (more exactly we should say ''second-order cyclostationary'').

Note that we work here with the covariance instead of the second-order moment. This is done to eliminate from the second-order statistics the contribution of the first order. Consider the following simple example, of great interest for stochastic resonance. Let x_t be the sum of a sinusoid $\sin(2\pi\nu t)$ and a stationary, zero mean noise b_t. Then, $M_x(t)$ is periodic and further, the second-order moment reads $E[x_t x_{t+\tau}] = \sin(2\pi\nu t)\sin[2\pi\nu(t+\tau)] + \Gamma_b(\tau)$. Therefore, the correlation is periodic and one can conclude that the process is cyclostationary. However, the covariance is equal to the covariance of the noise, and hence is not periodic. This shows that the cyclostationarity property of this signal is only due to its mean. Therefore, to understand the effect of cyclostationarity, we must work with cumulants instead of moments (the covariance is the second-order cumulant). This remark makes it possible to define an almost cyclostationarity up to order *n*. A stochastic process is said to be almost cyclostationary up to order *n* if its mean is almost periodic and if its cumulant functions $\mathrm{Cum}[x_t, x_{t+\tau_1}, \ldots, x_{t+\tau_i}]$ are almost periodic functions in *t* for $i = 1,\ldots,n-1$. As an example, the process $\sin(2\pi\nu t)+b_t$ described above is a first-order cyclostationary process.

Coming back to the covariance, if it is almost periodic, it admits a Fourier-Bohr series expansion in the variable t. Therefore, if we do a two-dimensional Fourier transform of the covariance ($\alpha\leftrightarrow t$, $\nu\leftrightarrow\tau$), we obtain a quantity which can be written as

$$S_x(\alpha,\nu)=\sum_{f_k\in A} s(f_k,\nu)\,\delta(\alpha-f_k),$$

where A is finite or infinite but countable. Function $S_x(\alpha,\nu)$ is discrete in α and continuous in ν. It is called the *spectral correlation*. Furthermore, frequency α is called a *cycle frequency*. The term ''spectral correlation'' comes from the fact that the spectral correlation examines the correlation between the harmonic components of x_t at frequencies ν and $\alpha-\nu$. It is well known that for a stationary process, this correlation is zero if the frequencies are different. Hence, for a stationary process, the spectral correlation is equal to zero, except for $\alpha=0$, for which $S_x(0,\nu)=s(0,\nu)$ reduces to the classical power spectrum. Furthermore, if the process is cyclostationary with fundamental period T, then we can show that

$$S_x(0,\nu)=\int\left[\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2}\Gamma_x(t,\tau)dt\right]e^{-2i\pi\nu\tau}d\tau.$$

This result illustrates what is usually done in the SR literature: averaging the covariance to define a ''stationarized'' spectrum. But in fact, this operation corresponds to taking the slice $\alpha=0$ in the spectral correlation. This explains the loss of information we mentioned earlier.

Finally, note that estimators of the spectral correlation exist [19], and therefore the spectral correlation is not only a theoretical tool but can also be used in practice.

III. REVISITING CLASSICAL SR WITH CYCLOSTATIONARY PROCESSES THEORY

The aim of this section is to examine stochastic resonance in static nonlinearities using the framework of cyclostationary processes.

We again insist on the fact that we will completely exploit the second-order statistics of the signal using the spectral correlation. Furthermore, the spectral correlation is a cumulant (covariance) and is therefore unpolluted by the mean of the signals.

We consider a deterministic almost-periodic signal s_t corrupted by a pure white noise b_t. The resulting signal y_t then passes through a static nonlinearity whose characteristic response is denoted by g. We consider both the additive and multiplicative corruption cases.

''Pure'' white noise means that for all integers n, the variables $b_{t_1},\ldots,b_{t_n}$ are statistically independent and identically distributed. Note that these restrictions make the noise strictly stationary. We will furthermore assume that the noise has zero mean, and that its probability density function $p_b(x)$ is even (this assumption can be easily eliminated and is adopted only for convenience). The variance of the noise is denoted by σ_b^2. In what follows, we need the cumulative density function (CDF) of the noise: It is defined as $F_b(x)=\int_{-\infty}^{x}p_b(y)dy$. Note that the evenness of $p_b(x)$ implies that $1-F_b(x)=F_b(-x)$.

A. SR for additive noise

In this section, we examine what happens to an almost-periodic signal s_t corrupted by an additive white noise b_t, when passing through a static nonlinearity. Let $y_t=s_t+b_t$ and $z_t=g(y_t)$. The mean of z_t is given by

$$M_z(t)=\int g(y)p_b(y-s_t)dy. \tag{1}$$

To evaluate the covariance function of z_t, we use the fact that b_t is a pure white noise: b_t and $b_{t+\tau}$ are statistically independent, and therefore y_t and $y_{t+\tau}$ are also independent. Any static nonlinearity preserves this independence; hence we obtain

$$\Gamma_z(t,\tau)=\mathrm{Cov}[z_t,z_{t+\tau}]=\mathrm{Cov}[z_t,z_t]\,\delta(\tau)=\mathrm{Var}[z_t]\,\delta(\tau). \tag{2}$$

The variance of z_t is then provided by

$$\mathrm{Var}[z_t]=\int g(y)^2p_b(y-s_t)dy-M_z(t)^2. \tag{3}$$

Note that Eqs. (1)–(3) show that, in general, the mean and the covariance function of z_t are periodic with respect to t. z_t is therefore in general cyclostationary, at least at order 2.

We cannot go further in this general case. To get an idea of what happens, we now turn to a simple nonlinearity.

Simple threshold device

Let us consider $g(x)=\mathbf{1}_{[\theta,+\infty[}(x)$ where $\theta\geqslant1$, and the periodic signal $s_t=\beta\sum_{i\in\mathbb{Z}}\mathbf{1}_{[-\eta/2,\eta/2]}(t-iT)$ where $0<\beta<1$ and $0<\eta\leqslant T/2$. $\mathbf{1}_I(x)$ stands for the characteristic function of interval I. Function s_t is even and admits the Fourier series decomposition $s_t=a_0/2+\sum_{k\geqslant1}a_k\cos(2\pi kt/T)$ with

$$a_0=\frac{2\eta\beta}{T},$$

$$a_k=\frac{2\beta}{\pi k}\sin\left(\frac{\pi k\eta}{T}\right).$$

According to Eq. (1), the mean of z_t reads $M_z(t)=\int_\theta^{+\infty}p_b(y-s_t)dy$ which leads to $M_z(t)=F_b(-\theta+s_t)$. Then, using Eq. (3), we have $\mathrm{Var}[z_t]=F_b(-\theta+s_t)-F_b(-\theta+s_t)^2$, which can be written as $\mathrm{Var}[z_t]=F_b(-\theta+s_t)F_b(\theta-s_t)$.

With the periodic signal considered, it is of interest that the mean and the variance of z_t have the same form as signal s_t, since they are a static nonlinear transformation of s_t.

Thus, the variance of z_t can be written as

$$\mathrm{Var}[z_t]=\beta_z\sum_{i\in\mathbb{Z}}\mathbf{1}_{[-\eta/2,\eta/2]}(t-iT)+F_b(-\theta)F_b(\theta),$$

with $\beta_z=F_b(-\theta+\beta)F_b(\theta-\beta)-F_b(-\theta)F_b(\theta)$. Therefore, the variance of the output admits the Fourier series expansion $\mathrm{Var}[z_t]=a_0^z/2+\sum_{k\geqslant1}a_k^z\cos(2\pi kt/T)$ where

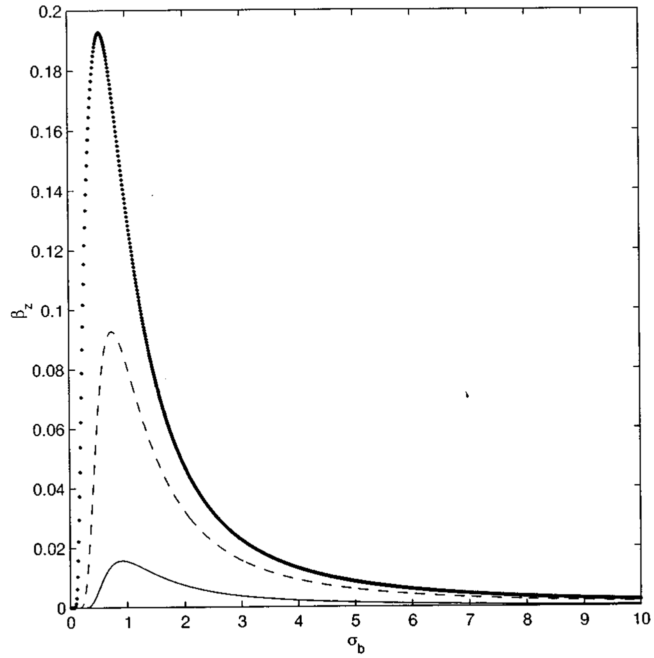

FIG. 1. Stochastic resonance in a threshold device attacked by a periodic signal additively corrupted by a Gaussian white noise. Amplitude β_z of the spectral correlation at a given cycle frequency plotted as a function of the standard deviation of the noise for $\theta = 1.2$ and $\beta = 0.1$: -, $\beta = 0.5$: - -, $\beta = 0.9$: $\cdots$. These curves have a maximum which reveals stochastic resonance (arbitrary units).

$$a_0^z = \frac{2\eta\beta_z}{T} + 2F_b(-\theta)F_b(\theta),$$

$$a_k^z = \frac{2\beta_z}{\pi k}\sin\left(\frac{\pi k\eta}{T}\right).$$

The spectral correlation of z_t readily follows: Since $\Gamma_z(t,\tau) = \mathrm{Var}[g(y_t)]\delta(\tau)$, after a two-dimensional Fourier transform, we obtain

$$S_z(\alpha,\nu) = \left(\frac{\eta\beta_z}{T} + F_b(-\theta)F_b(\theta)\right)\delta(\alpha) + \sum_{k\geq 1}\frac{\beta_z}{\pi k}\sin\left(\frac{\pi k\eta}{T}\right) \times\left[\delta\left(\alpha - \frac{k}{T}\right) + \delta\left(\alpha + \frac{k}{T}\right)\right].$$

Therefore, the spectral correlation of the output has a discrete spectrum of cycle frequencies, each of which supports a constant amplitude as a function of the harmonic frequency ν (even for $\alpha = 0$). Hence, the discrete spectrum usually seen in SR has disappeared. Nevertheless, the amplitude of the spectral correlation at a given nonzero cycle frequency presents the features of stochastic resonance, as shown in Fig. 1. In the graph, obtained for a Gaussian noise and for $\eta = T/2$, we show β_z as a function of σ_b, for some values of β and $\theta = 1.2$. The classical maximum characteristic of stochastic resonance is clearly seen. When the noise is very low, there is nearly no signal at the output, and the spectral correlation is very low. If the noise is very powerful, the signal crosses the threshold ''very randomly,'' and the cyclostationarity of the input is lost: The output tends to be stationary. Between these extremes, there is an optimal value of σ_b such that the output is the ''most'' cyclostationary.

We also observe in Fig. 1 that the strength of the effect decreases as β decreases. In the limit $\beta \to 0$, we obtain $\beta_z \to 0$, and the output is no longer cyclostationary. This is true since the input is stationary for $\beta = 0$ and the system is static. But for $\beta \to 0$, we obtain $S_z(0,\nu) = F_b(-\theta)F_b(\theta)$ which presents a maximum as a function of σ_b. This again shows that there is stochastic resonance, but for a stationary input. Thus, for threshold systems the term stochastic resonance is not the proper term. As proposed by Gammaitoni [8], the effect here should be called ''noise induced threshold crossings'' instead of stochastic resonance, because no matching condition between two time scales is needed. Indeed, we observe here that the effect is independent of the frequency we use. Nevertheless, we will continue to use the term stochastic resonance.

We now come back to the disappearance of the discrete spectrum in the variable ν. Usually, this discrete spectrum is observed when we work with the ''stationarized'' spectrum, i.e., the slice $\alpha = 0$ of the spectral correlation. However, the term $\mathcal{S}_z(0,\nu)$ does not present this discrete spectrum. This is because we work with cumulants rather than moments. To obtain the second-order moment, we must write $E[z_t z_{t+\tau}] = \Gamma_z(t,\tau) + E[z_t]E[z_{t+\tau}]$. Therefore, if we Fourier transform this expression with respect to both variables, we obtain the spectral correlation plus a term which will present a discrete spectrum for $\alpha = 0$. In this case, we would proceed as is usually done in the SR literature.

Instead, we first analyze the mean of the signal

$$M_z(t) = F_b(-\theta) + [F_b(-\theta+\beta) - F_b(-\theta)] \times\sum_{i\in\mathbb{Z}} \mathbf{1}_{[-\eta/2,\eta/2]}(t - iT),$$

which is periodic and whose Fourier series expansion reads

$$M_z(t) = \left(\frac{\eta[F_b(-\theta+\beta) - F_b(-\theta)]}{T} + F_b(-\theta)\right) + \sum_{k\geq 1}\frac{F_b(-\theta+\beta) - F_b(-\theta)}{\pi k}\sin\left(\frac{\pi k\eta}{T}\right) \times\left[\delta\left(\alpha - \frac{k}{T}\right) + \delta\left(\alpha + \frac{k}{T}\right)\right].$$

In Fig. 2, we plot the ratio $[F_b(-\theta+\beta) - F_b(-\theta)]/\sigma_b$ (amplitude of the fundamental) as a function of σ_b, for some values of β and $\theta = 1.2$ [note that the amplitudes of the harmonics are the same, except a factor of $\sin(\pi k\eta/T)/(k\pi)$]. We again observe SR here. This corresponds to the classical way of observing SR, since the above-mentioned ratio is roughly the signal-to-noise ratio (SNR) at the fundamental of the output. Note that this approach has been extensively used [9,10,4]. Chapeau-Blondeau [10] also studies the gain in terms of SNR between the input and the output at a given harmonic of the mean signal. The output SNR is defined as the ratio of the amplitude squared at a harmonic by the temporally averaged variance, which is given in our framework by $\mathcal{S}_z(0,\nu)$, since averaging is equivalent to considering the

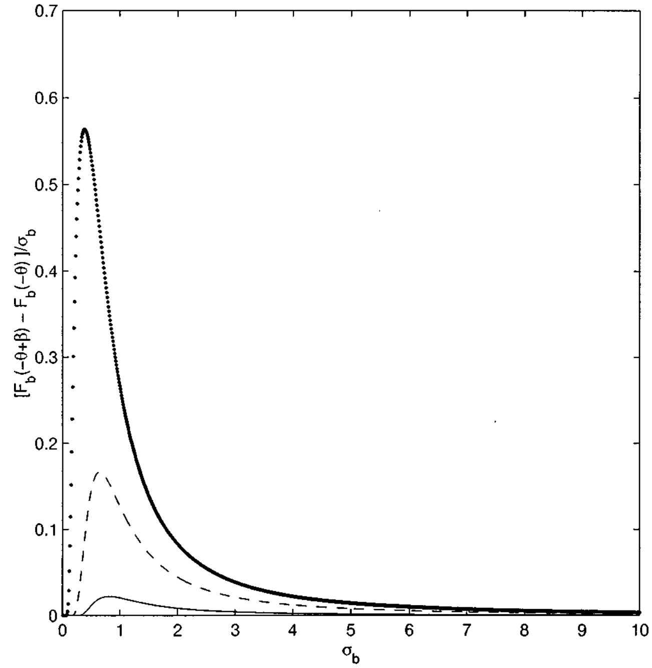

FIG. 2. Stochastic resonance in a threshold device attacked by a periodic signal additively corrupted by a Gaussian white noise. $[F_b(-\theta+\beta)-F_b(-\theta)]/\sigma_b$ plotted as a function of the standard deviation of the noise (''SNR'' for $E[z_t]$ at a given harmonic). $\theta=1.2$ and $\beta=0.1$, -; $\beta=0.5$, - -; $\beta=0.9$, $\cdots$ (arbitrary units).

$\alpha=0$ slice in the spectral correlation. Note also that gains (output SNR/input SNR) greater than one, as reported by Chapeau-Blondeau [10], have also been observed during the development of this work.

This discussion shows that the stochastic resonance effect studied here is a first-order effect, i.e., it is due to the fact that the mean of the input signal is periodic. Of interest is the observation of SR in the spectral correlation, which is a second-order statistical quantity ''unpolluted'' by the first-order statistics. This is a purely nonlinear effect.

We now turn to a second example of SR for deterministic signals, but for which the input signal is zero mean.

B. SR for multiplicative noise

Let s_t be an almost periodic deterministic signal, assumed to be strictly positive and of maximum amplitude lower than 1. Consider $y_t=s_t b_t$. Once again, b_t is a pure white noise with zero mean and variance σ_b^2. Signal y_t is then almost cyclostationary, and its covariance function reads $\Gamma_y(t,\tau)=\sigma_b^2 s_t^2\delta(\tau)$. Note that y_t is zero mean, so that the approach developed by Chapeau-Blondeau and Godivier [9,10] cannot be applied. The power spectrum of the input defined as the Fourier transform of the averaged covariance is flat: It does not present any peaks.

We study the output z_t of the simple threshold device $g(x)=\mathbf{1}_{[\theta,+\infty[}(x)$ attacked by y_t. The mean of z_t is found to be $E[z_t]=F_b(-\theta/s_t)$. Now, note that variables z_t and $z_{t+\tau}$ are statistically independent. Hence, the covariance of z_t reads $\Gamma_z(t,\tau)=\mathrm{Var}[z_t]\delta(\tau)$. The variance of z_t is given by $\mathrm{Var}[z_t]=F_b(-\theta/s_t)F_b(\theta/s_t)$. Hence, z_t is almost cyclostationary since its mean is almost periodic, as is its covariance. To obtain the spectral correlation, we then must perform the Fourier series decomposition of $F_b(-\theta/s_t)F_b(\theta/s_t)$, which is far from being an easy task.

We thus consider the following simple example: $s_t=\frac{1}{2}+\beta\Sigma_{i\in\mathbb{Z}}\mathbf{1}_{[-\eta/2,\eta/2]}(t-iT)$ where $0<\beta<\frac{1}{2}$ and $0<\eta\leqslant T/2$. Note that this function is never equal to zero. Furthermore, this function is even and admits the Fourier series decomposition $s_t=a_0/2+\Sigma_{k\geqslant 1}a_k\cos(2\pi kt/T)$ with

$$a_0=\frac{2\eta\beta}{T}+1,$$

$$a_k=\frac{2\beta}{\pi k}\sin\left(\frac{\pi k\eta}{T}\right).$$

Now, since s_t is constant by intervals, $F_b(-\theta/s_t)$ is also constant on the same intervals. In other words, $\mathrm{Var}[z_t]=F_b(-\theta/s_t)F_b(\theta/s_t)$ has the same structure as s_t and may be written $\mathrm{Var}[z_t]=F_b(-2\theta)F_b(2\theta)+\beta_z\Sigma_{i\in\mathbb{Z}}\mathbf{1}_{[-\eta/2,\eta/2]}(t-iT)$ with

$$\beta_z=F_b\left(-\frac{\theta}{1/2+\beta}\right)F_b\left(\frac{\theta}{1/2+\beta}\right)-F_b(-2\theta)F_b(2\theta).$$

Hence, the Fourier series expansion of $\mathrm{Var}[z_t]=F_b(-\theta/s_t)F_b(\theta/s_t)$ is $a_0^z/2+\Sigma_{k\geqslant 1}a_k^z\cos(2\pi kt/T)$ with

$$a_0^z=\frac{2\eta\beta_z}{T}+2F_b(-2\theta)F_b(2\theta),$$

$$a_k^z=\frac{2\beta_z}{\pi k}\sin\left(\frac{\pi k\eta}{T}\right).$$

Therefore, the spectral correlation of z_t reads

$$S_z(\alpha,\nu)=\left(\frac{\eta\beta_z}{T}+F_b(-2\theta)F_b(2\theta)\right)\delta(\alpha)+\sum_{k\geqslant 1}\frac{\beta_z}{\pi k}\sin\left(\frac{\pi k\eta}{T}\right)\left[\delta\left(\alpha-\frac{k}{T}\right)+\delta\left(\alpha+\frac{k}{T}\right)\right].$$

If we examine the evolution of β_z as a function of σ_b, we will observe the characteristic of stochastic resonance. However, in this case, since the input y_t has a periodic covariance, it is interesting to compare the spectral correlation of the output to the spectral correlation of the input.

Since $\Gamma_y(t,\tau)=\sigma_b^2 s_t^2\delta(\tau)$, the spectral correlation of y_t has the same structure as that of z_t, and reads

$$S_y(\alpha,\nu)=\sigma_b^2\left(\frac{\eta\beta_y}{T}+\frac{1}{4}\right)\delta(\alpha)+\sigma_b^2\sum_{k\geqslant 1}\frac{\beta_y}{\pi k}\sin\left(\frac{\pi k\eta}{T}\right)\left[\delta\left(\alpha-\frac{k}{T}\right)+\delta\left(\alpha+\frac{k}{T}\right)\right],$$

with $\beta_y=(\frac{1}{2}+\beta)^2-\frac{1}{4}$.

To compare $S_z(\alpha,\nu)$ and $S_y(\alpha,\nu)$ at cycle frequency k/T, it suffices to study the ratio $\beta_z/(\beta_y\sigma_b^2)$. We perform this comparison for the Gaussian case.

Figure 3 shows the evolution of $\beta_z/(\beta_y\sigma_b^2)$ as a function of the standard deviation of the noise σ_b in the Gaussian case. This figure is obtained for $\theta=1.2$ and for β

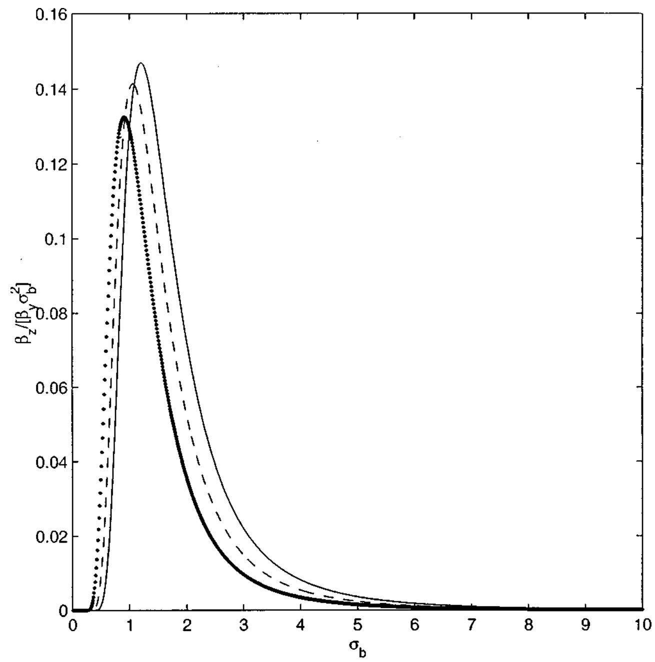

FIG. 3. Stochastic resonance in a threshold device attacked by a periodic signal multiplicatively corrupted by a Gaussian white noise ratio $\beta_z/(\beta_y\sigma_b^2)$ plotted as a function of the standard deviation of the noise (ratio between the amplitude of the spectral correlation before and after the nonlinearity). $\theta=1.2$ and $\beta=0.1$, -; $\beta=0.25$, - -; $\beta=0.45$, $\cdots$ (arbitrary units).

$=0.1, 0.25, 0.45$. Figure 4 represents the ratio $(\beta_z/\beta_y\sigma_b^2)$ as a function of θ and σ_b for $\beta=0.1$. These figures again express stochastic resonance, or to be more precise, as we mentioned earlier, ''noise induced threshold crossings.''

Finally, note again that we do not have a discrete spectrum, and the classical approach to SR would be to study $E[z_t]$, whose Fourier coefficients normalized by σ_b present the feature.

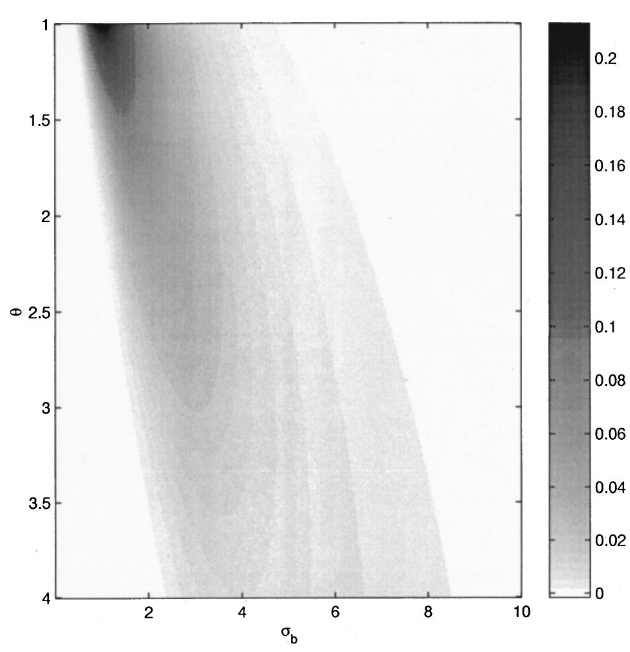

FIG. 4. Stochastic resonance in a threshold device attacked by a periodic signal multiplicatively corrupted by a Gaussian white noise. Two-dimensional graph of the ratio $\beta_z/(\beta_y\sigma_b^2)$ as a function of θ and σ_b for $\beta=0.1$ (arbitrary units).

To conclude this section, we see that the framework of cyclostationary processes offers an alternative view of stochastic resonance. It allows a complete description of the statistics of the output of the nonlinear system and provides a deeper understanding of SR than do traditional approaches. This understanding also uses cumulants rather than moments to decouple statistics of different orders. This can cause confusion since we have seen that the traditional power spectrum with peaks no longer exists: The periodicity in the output seems to be eliminated. In fact, this periodicity is clearly depicted by the structure of the spectral correlation that reflects the cyclostationarity of the output. Therefore, the framework proposed here goes farther than traditional approaches that actually only study the periodicity of the mean of the output signal.

To continue our development, we now consider stochastic resonance for stochastic processes additively corrupted by noise, and we then enter the field of ''aperiodic'' stochastic resonance. This kind of SR has already been considered [20,11,12,14,15,10] (see also [4], and references therein). In most of these works, the term aperiodic is quite ambiguous since signals are in general cyclostationary, and can be considered as special cases of communication signals. Furthermore, input-output measures are used to quantify stochastic resonance (e.g., coherence, transinformation). In the following sections, we study general communication signals and quantify stochastic resonance using only the output statistics of the system.

IV. COMMUNICATION SIGNALS AND STATIC NONLINEARITIES

In communication, a message to be transmitted is coded before emission. There exist many ways of coding, but basically, elements of the code are chosen in an alphabet of N letters $\{l_n,\ n=0,...,N-1\}$. To create a signal that is emitted, words are coded using letters in the alphabet, and each letter of the code is repeated during a period of T seconds. This leads to the simple form of a transmitted signal

$$s_t=\sum_{i=-\infty}^{+\infty} a_i f(t-iT), \tag{4}$$

where a_i is chosen in the alphabet $\mathcal{A}=\{l_n,\ n=0,...,N-1\}$. Function f may be ideally the characteristic function of interval $[0,T]$, but is generally more complicated. We will assume here that $f(t)$ is compactly supported over $[0,T]$.

Signal s_t is of course deterministic. However, the coding of a message leads to a very erratic sequence of a_i's. This sequence is therefore well described in a statistical sense. The following assumptions are usually made.

(1) a_i is a random variable that takes its values in the alphabet $\mathcal{A}=\{l_n,\ n=0,...,N-1\}$, each letter being chosen with equal probability $1/N$.

(2) The sequence $\{a_i,\ i=-\infty,\dots,+\infty\}$ is an independent and identically distributed sequence. In other words, the dis-

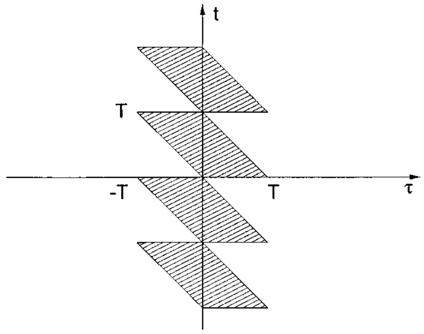

FIG. 5. Periodic support of communications signals considered in this paper. For t, τ outside the domain, s_t and $s_{t+\tau}$ are independent random variables. This property makes the calculation of the statistics of the output of a static nonlinearity easy (arbitrary units).

crete time signal a_i is a perfect white noise: if $i\neq j$, a_i and a_j follow the same law and are statistically independent.

(3) The random variable a_i is zero mean.

In this setting, signal s_t is a random signal. Signal s_t is a cyclostationary signal, with T as its fundamental period. This is easily verified since s_t is zero mean and

$$\Gamma_s(t,\tau)=E[s_t s_{t+\tau}]=\sigma_a^2[f(t)f(t+\tau)]*_t\sum_i \delta(t-iT),$$

where $*_t$ stands for convolution with respect to variable t, and σ_a^2 is the variance of variables a_i. The expression of the covariance function of s_t reveals the periodicity in t of the covariance.

For future calculations, it is worth examining the geometric structure of the support of the covariance. Since it is periodic, knowing the covariance for $t\in[0,T]$ is sufficient to knowing it for all t. Furthermore, since $f(t)$ is compactly supported, $\Gamma_s(t,\tau)$ is zero outside the domain defined by

$$\mathcal{D}_{t,\tau}=\begin{cases}0\leqslant t\leqslant T\\ -t\leqslant\tau\leqslant T-t.\end{cases} \tag{5}$$

The support of the covariance function is represented in Fig. 5. Note that this support is not only the support of the covariance, but is also the ''independence'' support. In other words, if (t,τ) is not in that support, the random variables s_t and $s_{t+\tau}$ are statistically independent.

We now consider a communication signal s_t corrupted by an additive pure white noise b_t. The corrupted signal is written as $y_t=s_t+b_t$. Since signal s_t is cyclostationary and b_t stationary, signal y_t is also cyclostationary, as demonstrated by the periodic structure of its covariance function $\Gamma_y(t,\tau)=\Gamma_s(t,\tau)+\sigma_b^2\delta(\tau)$. If we go into the spectral domain, the spectral correlation of y_t reads

$$\mathcal{S}_y(\alpha,\nu)=\sigma_a^2F(\nu)F(\alpha-\nu)(1/T)\Sigma_i\delta(\alpha-(i/T))+\sigma_b^2\delta(\alpha),$$

which shows that the contribution at nonzero cycle frequencies comes only from signal s_t. In other words, the stationary part of the signal appears only at the zero cycle frequency.

A. Output statistics for a static nonlinearity

In this section, we investigate the statistical properties of the output z_t of a static nonlinear filter attacked by $y_t=s_t+b_t$. Let $g(\,)$ denote the characteristic of the filter, so that $z_t=g(y_t)$.

Since y_t is cyclostationary, z_t will be in general cyclostationary. We assume here that z_t has the same fundamental period, that is T. We will come back to this assumption. If z_t is cyclostationary with period T, knowing its statistics for all t is equivalent to knowing them for $t\in[0,T]$. Therefore, we restrict t to that interval.

Mean

The mean of the output is given by $E[z_t]=E[g(y_t)]=\int g(y)p_{y},(y,t)dy$. Since s_t and b_t are independent variables, the PDF of their sum is the convolution product of their PDF. Furthermore, since $t\in[0,T]$, we write $s_t=af(t)$ where a is a random variable that takes its values in alphabet $\mathcal{A}=\{l_n,\ n=0,...,N-1\}$, each letter being equiprobable. Therefore, the PDF of y_t is written as $p_{y_t}(y,t)=1/N\Sigma_{n=0}^{N-1}p_b(y-l_nf(t))$. Hence, the mean of z_t is given by

$$E[z_t]=\int g(y)p_{y_t}(y,t)dy=\frac{1}{N}\sum_n\int g(y)p_b(y-l_nf(t))dy. \tag{6}$$

We note that this mean is zero if the nonlinearity characteristic is odd.

Covariance function

We evaluate $\Gamma_z(t,\tau)=E[z_tz_{t+\tau}]-E[z_t]E[z_{t+\tau}]$ for $t\in[0,T]$. The second-order moment is given by

$$E[z_tz_{t+\tau}]=\int g(y_1)g(y_2)p_{y_t,y_{t+\tau}}(y_1,y_2,t,\tau)dy_1dy_2$$

and we thus need the knowledge of the joint PDF of variables y_t and $y_{t+\tau}$ for $t\in[0,T]$. Two cases appear in this calculation.

(1) $\tau=0$: In this case, the statistics are completely determined by the PDF $p_{y_t}(y,t)$. We therefore obtain

$$E[z_t^2]=\frac{1}{N}\sum_n\int g(y)^2p_b(y-l_nf(t))dy \tag{7}$$

and the variance is obtained by subtracting from the expression the square of the mean.

(2) $\tau\neq0$: Here, the calculation depends on the belonging of $t+\tau$ to interval $[0,T]$.

(1) $t+\tau\notin[0,T]$: Since $t\in[0,T]$, since the noise is purely white and since the a_i's are independent and independent of the noise, variables z_t and $z_{t+\tau}$ are independent. The second-order moment then factorizes to give $E[z_tz_{t+\tau}]=E[z_t]E[z_{t+\tau}]$. The covariance is therefore equal to zero.

(2) $t+\tau\in[0,T]$. Since $t\in[0,T]$, we can write

$$y_t=f(t)a+b_t,$$
$$y_{t+\tau}=f(t+\tau)a+b_{t+\tau},$$

where a equals l_n with the probability $1/N$. Therefore, we obtain for the joint PDF of y_t and $y_{t+\tau}$

$$p_{y_t,y_{t+\tau}}(y_1,y_2,t,\tau)=\int p_{y_t,y_{t+\tau}/a=x}(y_1,y_2,t,\tau)p_a(x)dx$$
$$=\int p_{b_t}(y_1-f(t)x)p_{b_{t+\tau}}$$
$$\times(y_2-f(t+\tau)x)p_a(x)dx$$
$$=\frac{1}{N}\sum_n \int p_{b_t}(y_1-f(t)x)p_{b_{t+\tau}}$$
$$\times(y_2-f(t+\tau)x)\delta(x-l_n)dx$$
$$=\frac{1}{N}\sum_n p_b(y_1-f(t)l_n)p_b(y_2-f(t+\tau)l_n).$$

The second equality results from the independence between b_t and $b_{t+\tau}$, and the last one from $p_{b_t}=p_{b_{t+\tau}}=p_b$. The second-order moment then reads

$$E[z_tz_{t+\tau}]=\frac{1}{N}\sum_n \int g(y_1)g(y_2)p_b(y_1-f(t)l_n)$$
$$\times p_b(y_2-f(t+\tau)l_n)dy_1dy_2. \quad (8)$$

We finally obtain the covariance for $t\in[0,T]$ by subtracting from the expression the product of the first-order moments $E[z_t]E[z_{t+\tau}]$. This covariance is denoted as $\Gamma_z^{\mathcal{D}}(t,\tau)$.

The covariance for all t is obtained by periodizing the preceding result. We recall that the fundamental domain is given by

$$\mathcal{D}_{t,\tau}=\begin{cases}0\leqslant t\leqslant T\\ -t\leqslant\tau\leqslant T-t.\end{cases} \quad (9)$$

Assuming that $\Gamma_z^{\mathcal{D}}(t,\tau)$ is continuous at $\tau=0$, the covariance of z_t can be put in the form

$$\Gamma_z(t,\tau)=\mathbf{1}_{\mathcal{D}_{t,\tau}}(t,\tau)\Gamma_z^{\mathcal{D}}(t,\tau)*_t\sum_i \delta(t-iT)$$
$$+[\Gamma_z(t,0)-\Gamma_z^{\mathcal{D}}(t,0)]\delta(\tau). \quad (10)$$

The spectral correlation is then obtained by a two-dimensional Fourier transform. Instead of writing it explicitly, we will illustrate it in the examples in the next section.

B. Input-output statistics for a static nonlinearity

When examining a transformation of a random signal, it is of interest to study the input-output statistics for information on the transformation. For example, in the linear case, it is well known that the cross-correlation between the output and the input gives a method to identify the impulse response of the transfer. In the nonlinear case we are studying, there is of course no equivalent to the impulse response (or equivalently, a transfer function). However, input-output correlation may indicate how energy is transferred from the input to the output.

We are thus concerned with the quantity $\Gamma_{yz}(t,\tau)$ defined as the covariance between random variables y_t and $z_{t+\tau}=g(y_{t+\tau})$. Since y_t is assumed to be of zero mean, this covariance reduces to the second-order moment $E[y_tg(y_{t+\tau})]$.

Once again, we assume that the correlation is periodic of period T, so that we restrict t to be in $[0,T]$. Now, if $t+\tau$ is not in $[0,T]$, the variables are independent, the second-order moment factorizes, and it is therefore equal to zero since y_t is zero mean. For $\tau\neq 0$, if $t+\tau\in[0,T]$, we proceed as in the preceding section, and

$$\Gamma_{yz}^{\mathcal{D}}(t,\tau)=\frac{1}{N}\sum_n \int y_1g(y_2)p_b(y_1-f(t)l_n)$$
$$\times p_b(y_2-f(t+\tau)l_n)dy_1dy_2$$
$$=\frac{1}{N}\sum_n f(t)l_n\int g(y_2)p_b(y_2-f(t+\tau)l_n)dy_2. \quad (11)$$

For $\tau=0$, we obtain $\Gamma_{yz}(t,0)=1/N\Sigma_n\int yg(y)p_b(y-f(t)l_n)dy$. Then, the cross-correlation for all t is obtained by periodizing the previous result and reads

$$\Gamma_{yz}(t,\tau)=\mathbf{1}_{\mathcal{D}_{t,\tau}}(t,\tau)\Gamma_{yz}^{\mathcal{D}}(t,\tau)*_t\sum_i \delta(t-iT)$$
$$+[\Gamma_{yz}(t,0)-\Gamma_{yz}^{\mathcal{D}}(t,0)]\delta(\tau). \quad (12)$$

To evaluate the transfer of energy between different frequencies, the cross-correlation is then Fourier transformed to give the spectral cross-correlation.

Since the general theory presented above is difficult to interpret, we now turn to a specific example.

V. STOCHASTIC RESONANCE FOR COMMUNICATIONS SIGNALS IN A THRESHOLD DEVICE

In this section, we make explicit calculations of the spectral correlation of the output of static threshold devices for a two-state communications signal. Precisely, the alphabet used here is $\mathcal{A}=\{-1,+1\}$, and therefore we have $l_0=-1$ and $l_1=+1$.

Let θ be a positive real number greater than 1, and $g(x)=\mathbf{1}_{[\theta,+\infty[}(x)$. The output of such a nonlinearity is thus 1 when the output exceeds the threshold θ, and is 0 otherwise. Note that signal s_t alone cannot exceed the threshold since its maximum amplitude equals 1.

Using Eq. (6), the mean of z_t for $t\in[0,T]$ is expressed as

$$E[z_t]=\tfrac{1}{2}[F_b(-\theta-f(t))+F_b(-\theta+f(t))].$$

Equation (7) yields $E[z_t^2]=1/2[F_b(-\theta-f(t))+F_b(-\theta+f(t))]$. Thus, the variance of z_t is given by

$$\Gamma_z(t,0)=\tfrac{1}{2}[F_b(-\theta-f(t))+F_b(-\theta+f(t))]$$
$$\times\{1-[F_b(-\theta-f(t))+F_b(-\theta+f(t))]\}.$$

Now, Eq. (8) gives for $(t,\tau)\in\mathcal{D}_{t,\tau}$

$$E[z_tz_{t+\tau}]=\tfrac{1}{2}[F_b(-\theta-f(t))F_b(-\theta-f(t+\tau))$$
$$+F_b(-\theta+f(t))F_b(-\theta+f(t+\tau))].$$

We therefore obtain for $\Gamma_z^{\mathcal{D}}(t,\tau)$, after some algebra,

$$\Gamma_z^{\mathcal{D}}(t,\tau)=\tfrac{1}{4}[F_b(-\theta+f(t))-F_b(-\theta-f(t))]\times[F_b(-\theta+f(t+\tau))-F_b(-\theta-f(t+\tau))].$$

Finally, the covariance can be evaluated for all t using Eq. (10).

We can see that the periodicity of the covariance appears in the function $F_b(x)$ which is nonlinear. It will thus be very difficult in a general case to evaluate the spectral correlation of z_t. However, a simple case can be totally solved: $f(t)=\mathbf{1}_{[0,T]}(t)$. For this function, $f(t)=f(t+\tau)=1$, and the covariance reads, according to Eq. (10),

$$\Gamma_z(t,\tau)=\tfrac{1}{4}[F_b(-\theta+1)-F_b(-\theta-1)]^2\mathbf{1}_{\mathcal{D}_{t+\tau}}(t,\tau)*_t\times\sum_i\delta(t-iT)+\tfrac{1}{2}[F_b(-\theta-1)+F_b(-\theta+1)]\delta(\tau).$$

Let $H(\alpha,\nu)$ be the two-dimensional Fourier transform of $\mathbf{1}_{\mathcal{D}_{t,\tau}}(t,\tau)$. We then have

$$\mathcal{S}_z(\alpha,\nu)=\frac{1}{4}[F_b(-\theta+1)-F_b(-\theta-1)]^2\frac{1}{T}\sum_i H\left(\frac{i}{T},\nu\right)\delta\left(\alpha-\frac{i}{T}\right)+\frac{1}{2}[F_b(-\theta-1)+F_b(-\theta+1)]\delta(\alpha).$$

Hence, in this case, the amplitude of the spectral correlation is dependent on the cycle frequency k/T only through the term $H(k/T,\nu)$.

Gaussian noise

Let $e(x)=1/\sqrt{2\pi}\int_{-\infty}^{x}e^{-u^2/2}du$. Then, since the noise has a variance σ_b^2, the cumulative density function reads $F_b(x)=e(x/\sigma_b)$. When $f(t)$ is the characteristic function of $[0,T]$, the spectral correlation is given by

$$\mathcal{S}_z(\alpha,\nu)=\frac{1}{4}\left[e\left(\frac{-\theta+1}{\sigma_b}\right)-e\left(\frac{-\theta-1}{\sigma_b}\right)\right]^2\frac{1}{T}\times\sum_i H\left(\frac{i}{T},\nu\right)\delta\left(\alpha-\frac{i}{T}\right)+\frac{1}{2}\left[e\left(\frac{-\theta+1}{\sigma_b}\right)+e\left(\frac{-\theta-1}{\sigma_b}\right)\right]\delta(\alpha). \quad (13)$$

Figure 6 shows the amplitude of the spectral correlation at cycle frequency $1/T$ as a function of σ_b, omitting the factor $H(k/T,\nu)$. The graph is repeated for three values of θ: 1.01, 1.2, and 1.5. For a small noise standard deviation, the amplitude is very small. When the standard deviation is high, the amplitude is also small. In between, the amplitude of the cycle correlation passes through a maximum. This fact is interpreted as *stochastic resonance* or more precisely as noise induced threshold crossings. This can be understood as follows. When the noise is low, since the signal alone cannot exceed the threshold, the output will not efficiently reflect the cyclostationarity of the input. When the noise variance is high, the input signal passes the threshold ''very'' randomly, and the output is almost stationary. In between these extremes, there is an optimal variance of the noise for which the output is the most cyclostationary.

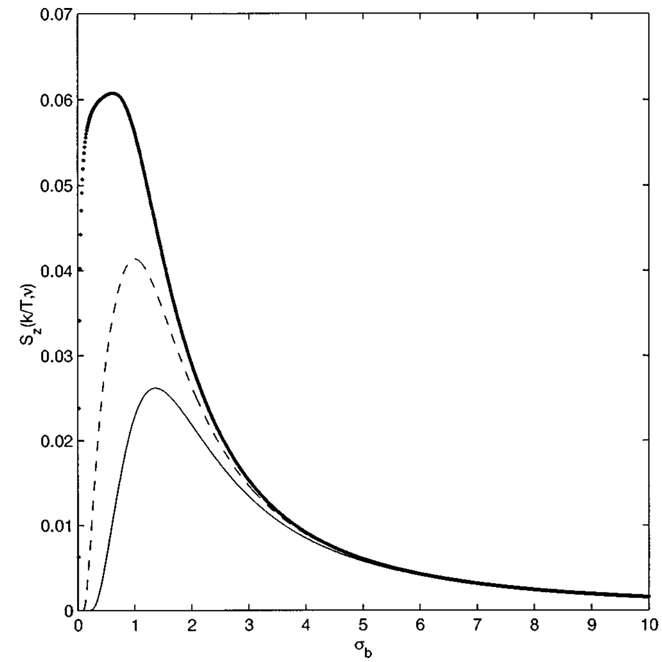

FIG. 6. Amplitude of the cycle correlation at cycle frequency k/T for the output of the simple threshold device attacked by a two-state communications signal corrupted by additive Gaussian noise. Function $f(t)$ is the characteristic function of $[0,T]$. $\theta=1.01$: $\cdots$, $\theta=1.2$: -, $\theta=1.5$: - - (arbitrary units).

The amplitude of the spectral correlation for the Gaussian case is quite low (see Fig. 6). However, the ratio of that amplitude to the amplitude at cycle frequency zero may not be so low. This is shown in Fig. 7, a two-dimensional graph of the ratio as a function of σ_b and θ. As can be seen in the figure, the lower the threshold, the greater the effect of SR; since the amplitude of the signal is 1, a small quantity of noise is needed to allow a crossing of the threshold, and the cyclostationarity of the input is nearly conserved. This ratio may then be interpreted as an index quantifying cyclostationarity. This interpretation comes from the fact that the zero cycle frequency corresponds to the stationary part of the signal, whereas nonzero cycle frequency reveals cyclostationarity. Therefore, the higher this ratio, the ''higher'' the cyclostationarity of the signal.

It is also interesting to study the behavior of the spectral correlation at the zero cycle frequency. We recall that $H(\alpha,\nu)$ is the two-dimensional Fourier transform of $\mathbf{1}_{\mathcal{D}_{t,\tau}}(t,\tau)=\mathbf{1}_{[0,T]}(t)\mathbf{1}_{[0,T]}(t+\tau)$. It is then easy to calculate explicitly $H(\alpha,\nu)$ which can be written $F(\nu)F(\alpha-\nu)$ with $F(\nu)=\exp(-i\pi\nu T)\sin(\pi\nu T)/(\pi\nu)$. Therefore, the spectral correlation at the zero cycle frequency reads [see Eq. (13)]

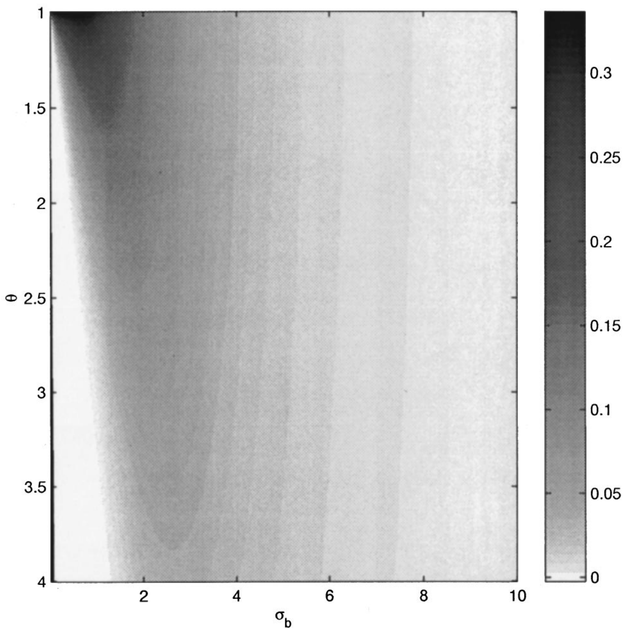

FIG. 7. Ratio of the amplitude of the spectral correlation at cycle frequency k/T over the amplitude at cycle frequency 0 for the output of the simple threshold device attacked by a two-state communications signal corrupted by additive Gaussian noise. The ratio is plotted as a function of σ_b and θ. Function $f(t)$ is the characteristic function of $[0,T]$ (arbitrary units).

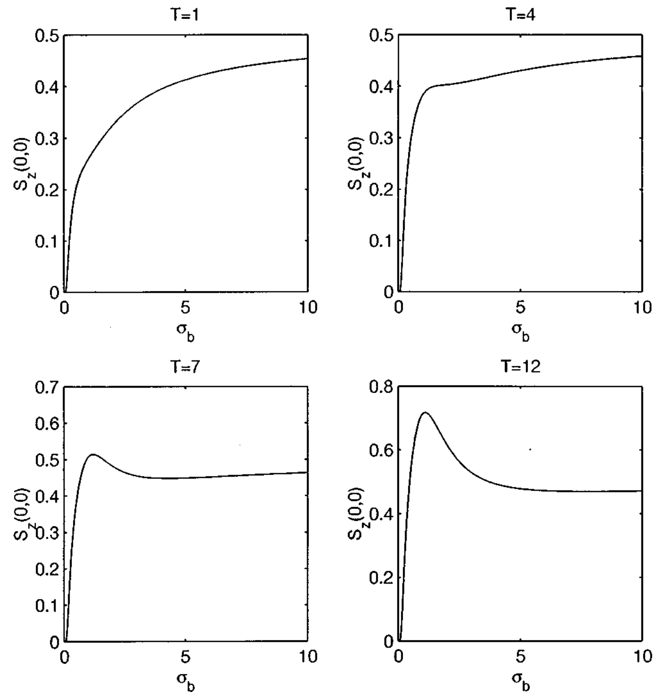

FIG. 8. Amplitude of the spectral correlation at cycle frequency 0 and $\nu=0$ for the output of the simple threshold device attacked by a two-state communications signal corrupted by additive Gaussian noise. Function $f(t)$ is the characteristic function of $[0,T]$. $\theta=1.2$. Period T takes different values on which the existence of a maximum depends (arbitrary units).

$$\mathcal{S}_z(0,\nu)=\frac{1}{4}\left[e\left(\frac{-\theta+1}{\sigma_b}\right)-e\left(\frac{-\theta-1}{\sigma_b}\right)\right]^2\frac{1}{T}\left|\frac{\sin(\pi\nu)}{\pi\nu}\right|^2+\frac{1}{2}\left[e\left(\frac{-\theta+1}{\sigma_b}\right)+e\left(\frac{-\theta-1}{\sigma_b}\right)\right].$$

The first term of the expression presents a maximum as a function of σ_b whereas the second does not. Therefore, it is not always sure that a maximum for $\mathcal{S}_z(0,\nu)$ as a function of σ_b exists; it depends on T. For $\theta=1.2$, we show in Fig. (8) $\mathcal{S}_z(0,0)$ plotted against σ_b for several values of period T. It appears that there exists a maximum only for sufficiently high T's. Since looking at the $\alpha=0$ slice in the spectral correlation is equivalent to looking at the ''stationarized'' spectrum, we see that we can miss the effect: this is not possible when looking at nonzero cycle frequencies.

To see the effect of stochastic resonance at the zero cycle frequency, one has to normalize the spectral correlation of the output by the spectral correlation of the input. For $\alpha=0$ and $\nu=0$, this ratio reads

$$\frac{\mathcal{S}_z(0,0)}{\mathcal{S}_y(0,0)}=\frac{(T/4)\{e[(-\theta+1)/\sigma_b]-e[(-\theta-1)/\sigma_b]\}^2+\frac{1}{2}\{e[(-\theta+1)/\sigma_b]+e[(-\theta-1)/\sigma_b]\}}{T+\sigma_b^2}.$$

This is shown in Fig. 9 where the effect is now seen. Note that the ratio can take values greater than one.

The limiting case $T\to 0$ is interesting. In this case, the input process y_t tends to be a pure white noise of PDF $p_{y_t}(y)=\frac{1}{2}[p_b(y-1)+p_b(y+1)]$. Therefore, this process is stationary, and we observe again ''noise induced threshold crossings'' for the output z_t. The effect is shown by the ratio

$$\frac{e[(-\theta+1)/\sigma_b]+e[(-\theta-1)/\sigma_b]}{2\sigma_b^2},$$

which can take values greater than one. The noise induced crossings for this limit case have also been studied with in-

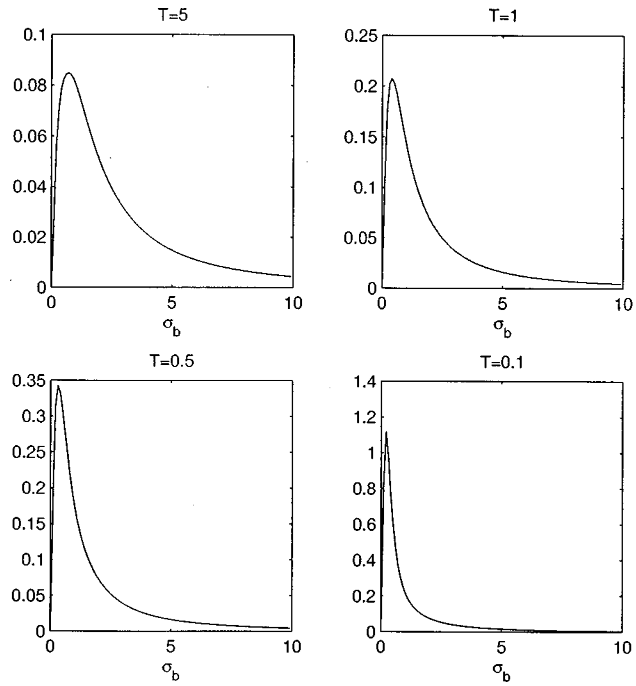

FIG. 9. Ratio of the amplitude of the spectral correlation at cycle frequency 0 and $\nu=0$ of the output to that of the input, in the case of the simple threshold device attacked by a two-state communications signal corrupted by additive Gaussian noise. Function $f(t)$ is the characteristic function of $[0,T]$. $\theta=1.2$. Stochastic resonance is now clearly seen (arbitrary units).

formation theory tools [15,16] where the effect is quantified using the input-output mutual information.

The cross-correlation between the input and the output is worth evaluating. After using Eqs. (11) and (12), we obtain, for $f(t)=\mathbf{1}_{[0,T]}(t)$,

$$\Gamma_{yz}(t,\tau)=\frac{1}{2}\left[e\left(\frac{-\theta+1}{\sigma_b}\right)-e\left(\frac{-\theta-1}{\sigma_b}\right)\right]\mathbf{1}_{\mathcal{D}_{t,\tau}}(t,\tau)*_t \times\sum_i \delta(t-iT)+\frac{1}{2}\left[\frac{\sigma_b e^{-(\theta-1)^2/2\sigma_b^2}}{\sqrt{2\pi}}+\frac{\sigma_b e^{-(\theta+1)^2/2\sigma_b^2}}{\sqrt{2\pi}}\right]\delta(\tau).$$

The spectral cross-correlation is the two-dimensional Fourier transform of the expression and reads in the Gaussian case considered

$$\mathcal{S}_{yz}(\alpha,\nu)=\frac{1}{2}\left[e\left(\frac{-\theta+1}{\sigma_b}\right)-e\left(\frac{-\theta-1}{\sigma_b}\right)\right]\frac{1}{T}\times\sum_i H\left(\frac{i}{T},\nu\right)\delta\left(\alpha-\frac{i}{T}\right)+\frac{1}{2}\left[\frac{\sigma_b e^{-(\theta-1)^2/2\sigma_b^2}}{\sqrt{2\pi}}+\frac{\sigma_b e^{-(\theta+1)^2/2\sigma_b^2}}{\sqrt{2\pi}}\right]\delta(\alpha).$$

This cross-correlation also reflects the stochastic resonance effect, since $\frac{1}{2}\{e[(-\theta+1)\sigma_b]-e[(-\theta-1)/\sigma_b]\}$ presents a maximum. Thus, this correlation can also be used to reveal the effect, and is close to the input-output coherence measure [12].

Furthermore, it can be shown that the cross spectral correlation reveals the correlation between frequency $\nu-\alpha$ of the input with the frequency ν of the output. It could therefore be used to understand how the energy is transferred from the input to the output.

VI. DISCUSSION AND CONCLUSION

We have shown that stochastic resonance can occur for cyclostationary stochastic processes. This fact has been demonstrated on widely used signals: communications signals.

To study the effect of SR, we work with the toolbox of cyclostationary stochastic processes. We have shown that this toolbox is appropriate to study SR. We have especially insisted on the fact that the averaging of the correlation function which is usually performed in the SR literature may lead to a loss of information. If performed on certain examples, the average of the correlation function may cause the SR effect to be overlooked (if we just look at the spectral density of the output). However, looking at nonzero cycle frequencies of the output, the effect is always revealed.

The importance of taking into account the entire spectral correlation (and the entire cross spectral correlation) also lies in the fact that the spectral correlation (and cross spectral correlation) quantifies the statistical interactions that may exist between frequencies of a signal. This is not the case for the classical spectrum (or cross spectrum) which assumes that frequencies are uncorrelated. Therefore, using cyclostationarity tools can provide more information concerning the physics of SR.

We note that the cyclostationarity of signals involved in stochastic resonance has been taken into account in some theories. For example, Jung and Hanggi [20] use Floquet-type solutions of the Fokker-Planck equation in order to evaluate correlation functions and spectral densities in the case of SR in the quartic potential. However, they again average in time to get ''stationarized'' quantities. Anyway, we believe that the extension from static nonlinearities to dynamical systems, as described in this paper, could be done using the Floquet theory.

We now examine the assumption used in Sec. IV A which states that the output signal of a static nonlinearity attacked by a cyclostationary signal is cyclostationary, with the same period of cyclostationarity. This assumption is true for most of the nonlinearities we can play with. However, some nonlinearities will make that assumption false. For example, a squarer may double the fundamental frequency and therefore halve the period. In that case, our calculations remain valid since if $T/2$ is a period of cyclostationarity, T is also a period. But some cases may be more troublesome. Consider a two-state communications signal (± 1) built with function $f(t)=\mathbf{1}_{[0,T]}(t)$. If this signal is squared, it is obviously not cyclostationary, since it is constant. But when corrupted by noise, it will be cyclostationary with period $T/2$. Therefore, the assumption can be viewed as a reasonable assumption.
The examples presented in this paper are very simple, since we restricted ourselves to the case of threshold devices. As already mentioned, we studied only noise induced threshold

crossings [8]. Several things must be done to show the importance of cyclostationarity in SR. The more important theoretically is to investigate SR for cyclostationary stochastic processes in dynamical systems (we are currently working in that direction for discrete time signals, and SR does occur). Finally, we would like to mention that communications systems often involve high nonlinearities, such as hard limiters to make decisions, and that SR for communications signals may be useful in practice.

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